Heterogeneous Star Celebrity Games*

Yen-Yu Chang[1], Chin-Chia Hsu[†1], and Ho-Lin Chen[3]

1 **Department of Electrical Engineering, National Taiwan University, Taipei, Taiwan**

3 **Corresponding author. Department of Electrical Engineering, National Taiwan University, Taipei, Taiwan,** `holinchen@ntu.edu.tw`

Abstract

With the growth of the Internet, social networks has recently received a lot of attention. The network creation game model was first proposed by Fabrikant et al. [6] to model the creation of social networks. Celebrity games is an extension of the network creation game model in which players try to minimize the total weight of players farther than a critical distance instead of the average distance to all other players.

In this paper, we study the problem of heterogeneous star celebrity games, which is an extension of several previous works [3, 2, 1]. We prove that the PoA is upper bounded by $O(\frac{n}{\beta})$ for all heterogeneous star celebrity games. The bound is asymptotically tight even when restricted to the max celebrity game model and matches with the upper bound on the star celebrity game model. We also show that this upper bound is tight for an extension of the bounded distance network creation games.

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1 Introduction

With the growth of the Internet, social networks has recently received a lot of attention. How do such networks arise? Instead of being controlled by a central authority, people have the freedom to choose their own friends and thus social networks are formed in a decentralized and non-cooperative way. Network creation games is a game theoretic model which aims to describe the formation of social networks.

The network creation game model was first proposed by Fabrikant et al. [6]. In this model, each player (node) can choose some players to establish direct connections and must pay for the costs. Once a connection is established, all players can use this connection in both directions even though this connection is established and paid by a single player. The goal for a player is to minimize the sum of connection costs and the average distance to all other players. In this model, each player treats all other players equally and must have connections to all other players. Many variations of this model have been studied [4, 5]. Most of these works focus on the price of anarchy (PoA) of network creation games, which is the ratio between the cost of the worst (pure-strategy) Nash equilbrium and the cost of social optimum.

Recently, a series of work focus on the variation with a critical distance. In this variation, the goal of a player is to have as many other players within the critical distance as possible

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† The first two authors make equal contribution to this paper and are listed in alphabetical order.

while keeping the connection cost low. Bilo et al. [3] proposed bounded distance network creation games, in which a player who maintains her (maximum or average) distance to the other nodes within a given bound incurs a cost equal to the number of activated edges; otherwise her cost is unbounded. Celebrity games [2] are extensions of the bounded distance network creation games. In this model, players have weights and the goal of a player is to minimize the sum of connection costs and the total weight of players who remain farther than the critical distance. Max celebrity games [1] is a variant of celebrity games in which the maximum weight of players who remain farther than the critical distance is considered instead of the total weight. Most of these works focus on star celebrity games in which the Nash equilibrium is a connected graph. Tight or near-tight bounds on PoA has been found for all of these models [3, 2, 1]

Our Results: In this paper, we study the problem of heterogeneous star celebrity games. In our model, the players try to minimize the sum of connection costs and penalties incurred from players farther than the critical distance. We allow the critical distance to be different for every player and the penalty function to be arbitrary non-negative functions. The penalty function does not even have to be monotone. Also, different players may have completely different penalty functions. This is a very natural problem to consider since in practice, people have very different preferences on their friends and making more friends does not always help. Our model is an extension of all previous models with critical distances [3, 2, 1].

In this paper, we prove that the PoA is upper bounded by $O(\sum_{i=1}^{n} \frac{1}{\beta_i})$ for all heterogeneous star celebrity games, where $\beta_i \geq 2$ is the critical distance for the i-th player. This bound simplifies to $O(\frac{n}{\beta})$ in the case which all players have the same critical distance β. In max celebrity game, there is a known matching lower bound $\Omega(\frac{n}{\beta})$ [1] and thus this upper bound cannot be further improved in general. The bound also matches with the best known upper bound on the star celebrity game model [2]. We also construct a matching lower bound example for the heterogeneous bounded distance network creation games.

2 Preliminaries

In this section, we formally describe our model. A *heterogeneous celebrity game* Γ is a tuple $\langle V, \alpha, \beta, f() \rangle$ where: $V = \{1, \dots, n\}$ is the set of players, $\alpha > 0$ is the cost of establishing a link, $\beta = (\beta_1, \beta_2, \dots, \beta_n)$ are the critical distances satisfying $2 \leq \beta_i \leq n-1$ for every i, and the function $f : V \times 2^V \to \mathbb{R}$ defines the penalty each player receives. $f(i, V_i)$ is the penalty which player i receives for having exactly the set of players V_i farther away from its critical distance β_i. The function f can be any non-negative function with $f(i, \phi) = 0$ (no penalty if all other players are within the critical distance) for all players i.

In game Γ, a strategy for a player u is a subset $S_u \subseteq V - \{u\}$, the set of players to whom player u pays for establishing direct links. For any strategy profile $S = (S_1, \dots, S_n)$, we define the *outcome graph* to be the undirected graph $G = (V, \{\{u, v\} | (u \in S_v) \vee (v \in S_u)\})$. Let $d_G(u, v)$ denote the distance (smallest number of links) between u and v in G. Notice that the outcome graph is undirected, thus even though a link is established by only one of the two players, once a link is established, it can be used by all players in both directions.

Given a strategy profile S and its corresponding outcome graph G. Let $V_u = \{v | d_G(u, v) > \beta_u\}$. If graph G is connected, the cost of a player u is denoted by

$$c_u(S) = \alpha |S_u| + f(u, V_u).$$

The cost of a player u has two components: the cost of establishing links is $\alpha |S_u|$ since the cost for establishing each link is α; the penalty caused by having nodes V_u with distance farther than his critical distance β_u is $f(u, V_u)$. The social cost of a strategy profile S is defined as

$$C(S) = \sum_{u \in V} c_u(S).$$

Let $\Gamma = \langle V, \alpha, \beta, f() \rangle$ be a heterogeneous celebrity game. A strategy profile S is a *Nash equilibrium* of Γ if no player has an incentive to deviate from his strategy. In other words, for each player u and each strategy S'_u, $C_u(S_{-u}, S'_u) \geq C_u(S)$. For any Nash equilibrium S, the corresponding outcome graph is called an *equilibrium graph*. A *star celebrity game* [2] is a celebrity game with all equilibrium graphs being connected and the star graph being both an equilibrium graph and the outcome graph of the optimal solution.

We use N(Γ) to represent the set of *pure-strategy Nash equilibria* of a game Γ. We use OPT(Γ) to denote the minimum social cost and OPT to denote the strategy profile with optimal social cost. The *price of anarchy (PoA)* of a heterogeneous celebrity game is the largest ratio between the cost of a Nash equilibrium of the game and the cost of OPT. More formally,

$$\texttt{PoA}(\Gamma) = \max_{S \in \texttt{N}(\Gamma)} \frac{C(S)}{\texttt{OPT}(\Gamma)}.$$

3 Tight Bounds on the Price of Anarchy

In this section, we will prove that the PoA of any heterogeneous star celebrity game is $O(\sum_{i=1}^{n} \frac{1}{\beta_i})$, which is asymptotically tight.

Given any Nash equilibrium, the following two lemmas bound the cost of player v by a deviation which connects v to all other nodes within distance β. If player v makes this deviation, the penalty reduces to 0 regardless of the penalty function f. Therefore, the derived upper bound holds for all non-negative penalty functions.

▶ **Lemma 1.** *Let N be a Nash equilibrium of a celebrity game $\Gamma = \langle V, \alpha, \beta, f() \rangle$ with graph G. If the subgraph G' obtained by removing player (node) v and all incident edges from G has k_v connected components, then $C_v(N) \leq (k_v + \frac{2n}{\beta_v})\alpha$.*

Proof. Choose one node from each connected component in G'. Let B_1 denote the set of all chosen nodes. Add the node v to G' and add an edge $\{u, v\}$ for every node $u \in B_1$. Call the resulting graph G''. Let B_i denote the set of nodes with distance exactly i from node v in G''. Let

$$D_i = B_i \cup B_{i+\beta} \cup B_{i+2\beta} \cup ...$$

For $i = 1, 2, 3, ..., \beta_v$. Let $|D_j|$ be the minimum among $|D_1|, |D_2|, ..., |D_\beta|$. $S_v = B_1 \cup D_j$ is a strategy with no penalty. Therefore, the cost of v is a most the cost of strategy S_v, which is $|B_1 \cup D_j|\alpha \leq (k_v + \frac{n}{\beta_v})\alpha$. ◀

▶ **Lemma 2.** *Given any connected undirected graph $G = (V, E)$. Let k_v be the number of connected components when node v and all its incident edges are removed from G, then*

$$\sum_{v \in V} k_v \leq 2n - 2.$$

Proof. Removing edges from G only increases k_i, so we only need to prove the case when G is a tree. In this case, if we remove a node v and all its incident edges, the remaining graph G' has exactly $\deg_G(v)$ connected components. Therefore

$$\sum_{v \in V} k_v = \sum_{v \in V} \deg_G(v) = 2n - 2$$

when G is a tree, which proves the lemma. ◀

Combining the above two lemmas, we get the main result of this paper, which is the following theorem.

▶ **Theorem 3.** *The PoA is at most $2 + \frac{n}{(n-1)} \sum_v \frac{1}{\beta_v}$ for any heterogeneous star celebrity game.*

Proof. From lemma 1, we know that any Nash equilibrium has cost at most $\sum_v (k_v + \frac{n}{\beta_v})\alpha \leq (2n - 2 + \sum_v \frac{1}{\beta_v})\alpha$. On the other hand, the optimal solution is the star graph. The total penalty for the star graph is 0 since all critical distances are at least 2. Therefore, the total cost of the optimal solution is the cost of constructing the edges, which is $(n-1)\alpha$. Combining the above, the PoA is at most $2 + \frac{n}{(n-1)} \sum_v \frac{1}{\beta_v}$. ◀

The above theorem shows that heterogeneous star celebrity games have PoA upper bound $O(\sum \frac{1}{\beta_v})$. When all players have the same critical distance β, this upper bound reduces to $O(\frac{n}{\beta})$. It has ben proved [1] that the upper bound is tight even for the max star celebrity game, which is a subset of heterogeneous star celebrity games. Our bound also matches with the upper bound [2] derived from the special case in which the penalty is the sum of the weights of the players which are farther away than β.

For the bounded distance network creation model in which the penalty is infinite whenever any node is farther away than β and all nodes having the same critical distances. It has been shown that better upper bounds exists [3]. However, if we extend the model to a heterogeneous bounded distance network creation game in which each player only needs to connect with a different subset of other players, the following theorem shows that the $O(\frac{n}{\beta})$ upper bound is also tight.

▶ **Theorem 4.** *There exists a heterogeneous bounded distance network creation game with PoA $= \Omega(\frac{n}{\beta})$.*

Proof. Prove by construction. Consider a game with $n = 2k\beta + 1$ players $A, B_{1,1}, B_{1,2}, \dots, B_{k,\beta}$ and $C_1, C_2, \dots, C_{k\beta}$. All players $B_{i,j}$ wants to stay within distance β to A. All players C_t wants to stay within distance β to all $B_{i,j}$.

Figure 1 is a Nash equilibrium of this game. The total cost is $\alpha(k^2\beta + k\beta) = \Theta(\frac{\alpha n^2}{\beta})$. On the other hand, for any $\beta \geq 2$, the optimal solution is a star graph which has total cost $(n-1)\alpha$. Therefore, the PoA of this game is $\Omega(\frac{n}{\beta})$.

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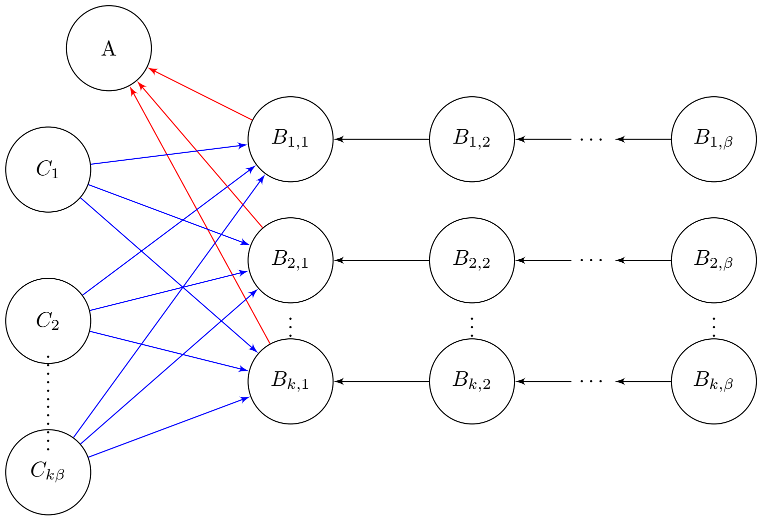

Figure 1 A equilibrium graph with cost $\Theta(\frac{\alpha n^2}{\beta})$

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